MS&E 319: Matching Theory	
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In this lecture we will see an algebraic approach toward analyzing matching in graph. In first section we will have a quick recap on properties of characteristic polynomials, then we will define matching polynomials. Finally, we will use interlacing families to construct Ramanujan graphs of all degrees.

1 Characteristic polynomial of a graph

Definition 1 (Characteristic polynomial) *Characteristic polynomial of a graph G with adjacency matrix A is defined as, $p(G,x) = det(xI - A)$.*

We now just quickly remind properties of characteristic polynomials. Here, let $\lambda_1 \geq ... \geq \lambda_n$ be eigenvalues of adjacency matrix A of graph G of size n. Let Δ be the maximum degree of graph.

1. $det(xI - A) = \prod(x - \lambda_i)$, where λ_i are eigenvalues of A.
2. If A is a symmetric matrix then its eigenvalues are real. Hence, characteristic polynomial of a graph are real-rooted.
3. $|\lambda_i| \leq \Delta$, $\forall i \leq n$.
4. (Exercise) $\sqrt{\Delta} \leq \max_{1\leq i\leq n}(\lambda_i) \leq \Delta$.
5. (Exercise) Let G be a tree then, $\max_{1\leq i\leq n}(\lambda_i) \leq 2\sqrt{\Delta - 1}$.

One way to connect combinatorial properties of adjacency matrix and its continuous nature is through the following formulae:

$$tr(A^k) = \sum_{i=1}^{n} \lambda_i^k.$$

Now, let a_is be coefficients of characterisitc polynomial, i.e. $det(xI - A) = \sum_{k=0}^{n} a_k x^k$. Then by first property we have $a_k = \sum_{S:|S|=n-k} (-1)^{n-k} \prod_{i\in S} \lambda_i$.

For any real-rooted polynomials we know that a_is are log-concave. Which means $a_k^2 \geq a_{k-1}a_{k+1}$.

2 Matching polynomial

Let $\mu(G,k)$ be the number of matchings of size k in graph G. We also define $\mu(G,0) = 1$.

Definition 2 *Define matching polynomial of graph G as $m(G,x) = \sum_{k=0}^{n/2} (-1)^k \mu(G,k) x^{n-2k}$.*

One reason that we define matching polynomial as above rather than simply defining its generating function ($\sum \mu(G,k)x^k$) is that historically physicistsused these polynomials to compute stability of molecules!!

Proposition 3 *Let u and v be any connected vertices in G. Let $G-u-v$ if made of G by removing u and v and $G-\{u,v\}$ is G without the edge between u and v. Then we have:*

$$\mu(G,k) = \mu(G-u-v,k-1) + \mu(G-\{u,v\},k)$$

Proof: In counting matchings either we use the edge between u and v and find $k-1$ other matchings from rest of the vertices or we do not use this edge at all. □

Corollary 4 $m(G,x) = m(G-\{u,v\},x) - m(G-u-v,x)$.

With similar analysis we have the following proposition.

Proposition 5 *If u is a vertex of the graph and $v_1,...,v_d$ are its neighbors then:*

$$\mu(G,k) = \mu(G-u,k-1) + \sum_{i=1}^{d} \mu(G-u-v_i,k-1)$$

Corollary 6 $m(G,x) = xm(G-u,x) - \sum_{i=1}^{d} m(G-u-v_i,x)$.

Proposition 7 *Let G_1 and G_2 be components of G then $m(G) = m(G_1)m(G_2)$*

Lemma 8 *If G is a forest then $det(xI-A) = m(G,x)$*

Proof: Since G is a forest there exist a node v with degree exactly 1. WLOG first node is adjacent only to second node. Then expand $det(xI-A)$ with respect to first row.

$$det(xI-A) = det \begin{bmatrix} x & 1 & 0 & \dots & 0 \\ 1 & x & \dots & & \\ 0 & \vdots & \ddots & & \\ \vdots & & & & \\ 0 & & & & \end{bmatrix}$$

$$= xdet(xI_{n-1} - A_{\{1\}}) - det(xI_{n-2} - A_{\{1,2\}}),$$

where A_S be the matrix made of A removing rows and columns whose indices are in S.

□

Corollary 9 *Matching polynomials of forests are real-rooted.*

In the rest of this section we want to prove that matching polynomials of all graphs are real-rooted. For this purpose we define path-tree graphs.

Definition 10 (path trees) *We call $T(G,v)$ the path-tree of graph G w.r.t v, where vertices of $T(G,v)$ are simple paths in G starting from v. Two vertices p_1 and p_2 are connected if corresponding path to p_1 is a prefix of p_2 and $|p_1| = |p_2| - 1$.*

Theorem 11 *Let G be a graph and u be any vertex in G then:*

$$\frac{m(G)}{m(G-u)} = \frac{m(T(G,u))}{m(T(G,u)-u)}.$$

Also, $m(G)$ divides $m(T(G,u))$.

Proof: We prove the theorem by induction on the number of vertices. Let $v_1, \ldots, v_d$ be neighbors of u in G. For simplicity, by T we mean $T(G, u)$. Note that T is a tree rooted in u, and u is connected to subtrees $T_1, \ldots, T_d$, where T_i is rooted on the path $u - v_i$. Now, we rewrite some identities by previous propositions.

1. $m(T) = xm(T-u) - \sum_{i=1}^{d} m(T-u-v_i)$, by Corollary 6.
2. $m(T-u) = \prod_{i=1}^{d} m(T_i)$, by Proposition 7.
3. $m(T-u-v_i) = m(T_i - v_i)\prod_{i\neq j} m(T_j)$, by Proposition 7.
4. $m(G) = xm(G-u) - \sum_{i=1}^{k} m(G-u-v_i)$ by Corollary 6.
5. $\frac{m(G-u)}{m(G-u-v_i)} = \frac{m(T_i)}{m(T_i-v_i)}$, by induction hypothesis.

Now, divide both sides of equation 4 by $m(G-u)$ and use equation 5 we have

$$\frac{m(G)}{m(G-u)} = x - \sum \frac{m(T_i)}{m(T_i - v_i)}.$$

Also, divide both sides of equation 1 by $m(T-u)$ and use equation 2 and 3.

$$\frac{m(T)}{m(T-u)} = x - \sum \frac{m(T-u-v_i)}{\prod_{i=1}^{d} m(T_i)} = x - \sum \frac{m(T_i)}{m(T_i - v_i)}.$$

So, $\frac{m(G)}{m(G-u)} = \frac{m(T)}{m(T-u)}$. Now, to prove that $m(G)$ divides $m(T)$ we again use induction and use the fact that $\frac{m(T)}{m(G)} = \frac{m(T-u)}{m(G-u)}$. RHS is a simple polynomial by induction hypothesis so the proof is complete.

□

Corollary 12 *All the roots of $m(G)$ are real and they are at most $2\sqrt{\Delta - 1}$.*

Corollary 13

$$\mu(G,k) \geq \mu(G,k-1)\mu(G,k+1)$$

One can show that finding maximum root of matching polynomial of G is related to finding number of closed walks of in path-tree graph of G.

Problem 1 (Open) *Give a PTAS for counting the number of closed walks of length k, starting from u in $T(G,u)$.*

3 Constructing bipartite Ramanujan graphs

Informally Ramanujan graphs are best possible expanders. Formally we will define them as follow:

Definition 14 (Ramanujan Graphs) *G is Ramanujan if it is $d-$regular graph and*

$$\max_{|\lambda|<d} |\lambda| \leq 2\sqrt{d-1}.$$

In fact, $2\sqrt{d-1}$ is the best possible upper bound we can get. It was shown by Alon and Boppana that for and $\epsilon > 0$ and for sufficiently large n, non-trivial eigenvalue of $d-$regular graphs of size n has value $\geq 2\sqrt{d-1} - 1$.

We will breifly go over the proof of Marcus, Spielman and Srivastava on the existence of infinte sequence of bipartite Ramanujan graphs.

Theorem 15 *For every $d \geq 3$, there exists an infinite sequence of $d-$regular bipartite Ramanujan graphs.*

Proof: Given a Ramanujan graph G we will double its size by using a process called 2-lift process, and we will show that the graph remains Ramanujan after the amplification.

Definition 16 (2-lift process) *Given a graph $G(V, E)$ construct graph H where vertices of H are 2 copies of vertices of G, call it V and V'. For each edge (u, v) in G with probability $1/2$ we add edges (u, v) and (u', v') (parallel edges) and with probability $1/2$ we add edges (u, v') and (u', v) (crossing edges).*

Let S correspond to a realization of 2-lift process define $S(u, v) = 1$ if u, v are parallel in H and -1 otherwise. Also, define A_s be the matrix which is zero on non-edges and is equal to $S(u, v)$ on edges. Let A_{new} be adjacency matrix of graph after 2-lift process. And let $\lambda(\mathbf{M})$ be the set of eigenvalues of $\mathbf{M}$. We will use the following fact without proof.

$$\lambda(A_{new}) = \lambda(A) \cup \lambda(A_S).$$

Note that maximum eigenvalue of A_s is maximum root of $\chi_{A_S}(x) = det(xI - A_S)$. We will prove that there exist realization of $S \in \{-1, 1\}^n$ so that $maxroot(\chi_{A_S}(x)) \leq maxroot(\mathbb{E}_{s\sim\{-1,1\}^n}[\chi_{A_S}(x)])$. Also, we will show $\mathbb{E}_{s\sim\{-1,1\}^n}[\chi_{A_S}(x)] = m(G)$. Then by corollary 12 we know that $maxroot(m(G))$ is at most $2\sqrt{d-1}$, and we are done.

First we show that $\mathbb{E}_{s\sim\{-1,1\}^n}[\chi_{A_S}(x)] = m(G)$. By definition using permutation definition of determinant we have

$$\mathbb{E}_{s\sim\{-1,1\}^n}[\chi_{A_S}(x)] = \sum_{\sigma} (-1)^{sgn(\sigma)} x^{\#\textit{fixed points}} \mathbb{E}_S[\prod A_S(i, \sigma(i))] = m(G).$$

Second equality is true because we know $\mathbb{E}(A_S(u, v)) = 0$ where u, v is an edge, so in the product above we are just left with cycles if length 2 or matchings of size $(n - \#\textit{fixed points})/2$.

Now, to prove $maxroot(\chi_{A_S}(x)) \leq maxroot(\mathbb{E}_{s\sim\{-1,1\}^n}[\chi_{A_S}(x)])$ we define interlacing families.

Definition 17 (Polynomials with common interlacing) *We say two polynomials $p(x)$ and $q(x)$ have common interlacing if their degree differs at most 1 and if α_i is i-th root of $p(x)$ and β_i is i-th root of q then we have $\beta_{i-1} \leq \alpha_i \leq \beta_i$ and $\alpha_i \leq \beta_i \leq \alpha_{i+1}$.*

Definition 18 (Interlacing family) *We say $\{P_S\}_{S\in\{-1,1\}^m}$ is an interlacing family if they can be placed on leaves of tree, so that when every internal node is the average of its direct descendants then sets of siblings have common interlacing.*

We use the following theorem without proof.

Theorem 19 *Let $p(x)$, and $q(x)$ be monic real-rooted polynomials. Then they have a common interlacing if and only if $\lambda p(x) + (1-\lambda)q(x)$ is real-rooted for all $\lambda \in [0,1]$.*

So if we prove that $P_{S_1,\ldots,s_k}(x) = \mathbb{E}_{s_{k+1},..,s_n}[\chi_{s_1,\ldots,s_n}(x)]$ are interlacing families, then we can start from the root $m(G)$ and derandomize toward the descendant that have the smaller maximum root until we get to a leaf.

By Theorem 19 we simply need to show that for all $k \leq n$, and all assignments of s_i, where $i \leq k$, the polynomial $\lambda P_{S_1,\ldots,s_k,1}(x) + (1-\lambda)P_{S_1,\ldots,s_k,-1}(x)$ is real-rooted for all $\lambda \in [0,1]$.

We will prove a stronger statement by showing that $\mathbb{E}_{s_{k+1},..,s_n}[\chi_{s_1,\ldots,s_n}(x)]$ is real-rooted for every independent binomial distribution on $s_1,\ldots,s_k$. This is the same as proving $\mathbb{E}_s[P_s(x-d)]$ is real-rooted. This is useful because we can write $A_S + dI$ as sum of familiar rank one matrices as follow.

Let δ_u be the indicator vector of vertex $u \in V$.

$$\delta_u(v) = \begin{cases} 1 & u = v \\ 0 & o.w. \end{cases}$$

Also, for each edge (u,v) define:

$$r_{(u,v)} = \begin{cases} \delta_u - \delta_v & \textit{with probability } \lambda_{(u,v)} \\ \delta_u + \delta_v & \textit{with probability } 1-\lambda_{(u,v)} \end{cases}$$

Then we have:

$$A_s = \sum_{u\sim v} r_{(u,v)} r_{(u,v)}^T - dI.$$

Hence we have,

$$\mathbb{E}_s[P_s(x)] = \mathbb{E}_s[det(xI - (dI + A_s)] = \mathbb{E}_s[det(xI - \sum_{u\sim v} r_{(u,v)} r_{(u,v)}^T].$$

This polynomial is real-rooted using real-stability properties of polynomials. The proof idea is that we can view this univariate polynomial as a restriction of a multivariate polynomial, and we will show that it is a transformation of a real-stable polynomial that preserves stability.

Let $B_i = \mathbb{E}[r_i r_i^T]$. Then,

$$\mathbb{E}_s[det(xI - \sum_{u\sim v} r_{(u,v)} r_{(u,v)}^T] = \prod_{i=1}^{m}(1 - \frac{\partial}{\partial z_i}) det(xI + \sum_{i=1}^{m} z_i B_i)).$$

And we know $(1 - \frac{\partial}{\partial z_i})$ preserves real-stability. So, we showed $P_{s_1,\ldots,s_k}$ are interlacing families and the proof is complete. □

References

[1] A. Marcus, D. A. Spielman, and N. Srivastava. *Interlacing families i: Bipartite ramanujan graphs of all degrees.* In Foundations of Computer Science (FOCS), 2013 IEEE 54th Annual Symposium on, pages 529537, Oct 2013.