MS&E 319: Matching Theory
Homework 1
Due: May 1, 2019

Problem 1. Prove that the edges of a bipartite graph with maximum degree δ can be colored with δ colors such that no two edges that share a vertex have the same color.

Problem 2. A square matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{n \times n}$ is doubly stochastic if the entries of the matrix are nonnegative, and the sum of entries in every row and column is equal to one. The Birkhoff-von Neumann theorem states that one can write any doubly stochastic matrix as a convex combination of permutation matrices. Prove this theorem and show that we can write any doubly stochastic matrix as a convex combination of at most $n^2 - n$ permutation matrices.

Problem 3. We have the following Theorem:

Theorem 1 *Let G be a graph (not necessarily bipartite) and let M be a matching in G and let B be a blossom with respect to M. Then M is a maximum size matching in G if and only if M/B is a maximum size matching in G/B.*

Give an example of a graph G, a matching M and a blossom B for M such that a maximum matching M^* in G/B does not lead to a maximum matching in G. Explain why this does not contradict Theorem 1.

Problem 4. (extra credit) A graph $G = (V, E)$ is said to be *factor-critical* if, for ll $v \in V$, we have that $G \setminus \{v\}$ contains a perfect matching. In parts (a) and (b) below, G is a factor critical graph.

1. Let U be *any* minimizer in the Tutte-Berge formula for G. Prove that $U = \emptyset$.
2. Deduce that when Edmonds algorithm terminates the final graph (obtained from G by shrinking flowers) must be a single vertex.
3. Given a graph $H = (V, E)$, an *ear* is a path $v_0 - v_1 - v_2 - \ldots - v_k$ whose endpoints (v_0 and v_k) are in V and whose internal vertices (v_i for $1 \leq i \leq k-1$) are not in V. We allow that v_0 be equal to v_k, in which case the path would reduce to a cycle. Adding (a 'trivial' ear) simply means adding an edge to H. An ear is called *odd* if k is odd, and even otherwise; for example, a trivial ear is odd.
   (a) Let G be a graph that can be constructed by starting from an odd cycle and repeatedly adding odd ears. Prove that G is factor-critical.
   (b) Prove the converse that any factor-critical graph can be build by starting from an odd cycle and repeatedly adding odd ears.

Problem 5. A *stable set* S (sometimes, it is called also an independent set) in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in S. If we let P denote the convex hull of all (incidence vectors of) stable sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for P.

1. Give a graph G for which P is *not* equal to

$$\begin{aligned} \{x \in \mathbb{R}^{|V|} : x_i + x_j &\leq 1 && \text{for all } (i,j) \in E \\ x_i &\geq 0 && \text{for all } i \in V\} \end{aligned}$$

2. Show that if the graph G is bipartite then P equals

$$\begin{aligned} \{x \in \mathbb{R}^{|V|} : x_i + x_j &\leq 1 && \text{for all } (i,j) \in E \\ x_i &\geq 0 && \text{for all } i \in V\}. \end{aligned}$$