MS&E 319: Matching Theory
Homework 3
Due: June 10, 2019

Problem 1. Consider a graph G of size n with adjacency matrix A. Let $p(G,x) = \det(xI - A)$ be the characteristic polynomial of G with roots $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Let Δ be the maximum degree of G.

1. Show that
$$\sqrt{\Delta} \leq \lambda_1 \leq \Delta.$$
2. Prove that if G is a tree, then
$$\lambda_1 \leq 2\sqrt{\Delta - 1}.$$

Problem 2. Prove the rural hospital theorem for stable matchings. More precisely, show that if the preferences of men and women are strict, then the set of men and women who are matched is the same in all stable matchings.

Problem 3. For any two matchings μ and μ', define the join operator $\mu \vee \mu'$ as a function that assigns to each man m his more preferred option of the two matches $\mu(m), \mu'(m)$. Similarly, define the meet operator $\mu \wedge \mu'$ to be the function that assigns to each man his less preferred option. Prove that if μ and μ' are stable matchings, then so are $\mu \vee \mu'$ and $\mu \wedge \mu'$.