MS&E 319: Matching Theory
Instructor: Amin Saberi
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In this lecture we first describe combinatorial approaches to finding a perfect matching. Next we introduce the LP formulation and the polyhedral description of matchings and their connection.

1 Classic Combinatorial Results

First recap the Hall's theorem that gives us a necessary and sufficient condition to see whether a bipartite graph has a perfect matching.

Theorem 1 (Hall's Theorem) *A bipartite graph $G(U, V, E)$, where $|U| = |V| = n$ has a perfect matching if and only if:*

$$\forall S \subseteq U, \quad |N(S)| \geq |S|.$$

where $N(S)$ is the set of neighbors of vertices of S in V.

Definition 2 (Augmenting Path) *A path $v_1, v_2, \ldots, v_{2k+1}$ is an augmenting path in $G(V, E)$ w.r.t matching M, if and only if:*

- *$(v_i, v_{i+1}) \in E$, $1 \leq i \leq 2k$, and*
- *$(v_{2i}, v_{2i+1}) \in M$, $1 \leq i \leq k$, and v_1, and v_{2k+1} are not matched.*

Proposition 3 *Either M is a maximum matching or there exists an augmenting path in G with respect to M.*

Proof: Suppose N is another matching, such that $|N| > |M|$. Consider the symmetric difference of N and M. Each vertex is of degree 2, so the resulting graph is a union of even cycles and paths. But, there are the same number of edges from M and N in each cycle and even-length path. Since, $|N| > |M|$ there must exist an odd path, such that its starting and ending edges are in N. This is an augmenting path w.r.t M. □

By the previous proposition, to find a maximum matching in a bipartite graph it is enough to start with any matching and iteratively find an augmenting path in our graph and increase the size of our matching by adding odd edges of the augmenting path to the matching and removing even edges from it. Now, the question is how can we efficiently find augmenting paths?

Algorithm 1: Find an augmenting path

Direct edges in M from U to V;
Direct edges in $E \setminus M$ from V to U;
Add vertices s, t. Connect s to all unmatched vertices of V. Connect all unmatched vertices of U to t;
Run BFS algorithm (find shortest path from s to t).

By construction of the directed graph in the algorithm, if there exists a path from s to t, there exists an augmenting path. Looking at the tree produced at the last step of algorithm one can see matchings and vertex covers are related in an interesting way.

Definition 4 (Vertex Cover) *A set $S \subseteq V$ is a vertex cover if for every $(u, v) \in E$, $u \in S$ or $v \in S$.*

Lemma 5 *If M is a matching and S is vertex cover then $|S| \geq |M|$.*

Proof: For a given vertex cover S, at least one side of each of M must be in S. So $|S| \geq |M|$. □

Theorem 6 *If M^* is a maximum matching in a bipartite graph G and S^* is a min vertex cover in G then $|M^*| = |S^*|$.*

We leave the proof of this theorem as an exercise (Hint: use the tree found by the BFS in the augmenting path algorithm).

2 Polyhedral Approaches

Assume the graph has a perfect matching. Can we find the matching by solving a linear programming relaxation of the problem? Given a graph $G(V, E)$ and a matching M, let $\mathbf{X}^M \in \mathbb{R}^{|E|}$ be the indicator vector of matching M:

$$X_e^M = \begin{cases} 1 & e \in M \\ 0 & otherwise \end{cases}.$$

For a subset of vertices S, define the boundary as $\delta(S) = \{e = (u, v) | e \in E, |\{u, v\} \cap S| = 1\}$. With abuse of notation we define $\delta(u)$ to be the set of edges incident to u.

2.1 Bipartite Graphs:

LP relaxation of matching problem in bipartite graphs can be written as:

$$\begin{aligned} \text{maximize} \sum_{e \in E} X_e w_e & & (1) \\ \text{s.t.} \sum_{e \in \delta(v)} X_e = 1 & \qquad \forall v \in V \\ X_e \geq 0 & \qquad \forall e \in E \end{aligned}$$

Define the set P_M as convex hull of X^M for all perfect matchings and P_{LP} as the set of all possible fractional matchings. Formally:

$$P_M = \text{conv}\{\mathbf{X}^M | M \text{ is a perfect matching.}\}$$

$$P_{LP} = \{\mathbf{X} | X_e \geq 0 \ \forall e \in E, \sum_{e \in \delta(v)} X_e = 1 \ \forall v \in V\}$$

The following claim proves that all the end points of polytope P_{LP} are integral.

Claim 7 *If G is a bipartite graph then $P_{LP} = P_M$.*

Proof: The first direction is easy. Since each vector $\mathbf{X}^M$ satisfies matching condition ($\sum_{e\in\delta(v)} X_e = 1$), any convex combination of these vectors also satisfies the matching condition. Thus, $P_M \subseteq P_{LP}$.

For the second direction, we use proof by contradiction. Suppose $P_{LP} \nsubseteq P_M$. Then P_{LP} must have a corner point that cannot be written as convex combination of perfect matchings. Let X be such a corner point. Define

$$F = \{e \in E | 0 < x_e < 1\}.$$

If F is the empty set, then x is an integral perfect matfching. Otherwise, F must contain a cycle of even length (since it's bipartite). Let C be such an even cycle and ϵ be the minimum edge value in this cycle. Let $d = (..., \epsilon, ..., -\epsilon, ..., \epsilon, ..., -\epsilon)$ where ϵ and $-\epsilon$ appear in position of the edges in C and the sign depends on the parity of the edge's appearance around the cycle. One can see that $x + d$ and $x - d$ are both feasible solutions of P_{LM} and x can be written as a convex combination of $x + d$ and $x - d$. This contradicts x being a corner point of P_{LM}.

$\square$

2.2 Market Equilibrium Prices:

In this section we look at dual problem of LP 1. Let $G(U, V, E)$ be the graph where U is the set of buyers and V is the set of sellers. The primal LP formulation for maximum matching is:

$$\begin{aligned}
\text{maximize } & \sum_{e\in E} X_e w_e & \\
\text{s.t. } & \sum_{e\in\delta(v)} X_e \leq 1 & \forall v \in V \\
& \sum_{e\in\delta(u)} X_e \leq 1 & \forall u \in U \\
& X_e \geq 0 & \forall e \in E
\end{aligned}$$

Assign S_i to constraints for vertices in U, and P_j for vertices in V, we got the following dual problem:

$$\begin{aligned}
\text{minimize } & \sum_{v\in V} P_v + \sum_{u\in U} S_u & \\
s.t. \ & P_v + S_u \geq w_{(u,v)} & \forall (u, v) \in E \\
& P_v, S_u \geq 0 &
\end{aligned}$$

In the optimal matching, if buyer u is assigned to seller v, in other words $X_{u,v} > 0$, by complimentary slackness condition we must have $S_u + P_v = w_{(u,v)}$. Now, the question is what are the market equilibrium prices (P_vs) that maximize the social welfare. The following auction algorithm can do the work.

Algorithm 2: Finding equilibrium prices

Start with prices at 0;
while *True* **do**
    Construct equality graph G in which $u \sim v$ if $v \in \text{argmax}_k\{w_{u,k} - P_k\}$;
    Find maximum matching M in G;
    if *G has perfect matching* **then**
        We are done! Return the prices;
    else
        Find set $S \in U$ s.t. $|N(S)| < |S|$ (this set exists because of Hall's theorem);
        Increase prices of $N(S)$ by a small constant c;
        To avoid infinite prices, decrease all prices until minimum price is 0;
    end
end

Claim 8 *Algorithm 2 terminates.*

Proof: Consider the objective value $\sum_i P_i + \sum_j S_j$. At each step we are decreasing this amount by at least $c(|S| - |N(S)|)$, so in general the amount $\sum_i P_i + \sum_j S_j$ decreases at each step, until we hit the optimal value. □

2.3 General Graphs:

LP 1 does not have integer solution for general graphs. Consider a triangle with $X_e = 1/2$ for all edges.

Exercise: Prove that corner points of P_{LP} for general graphs are always half integral.

To get rid of triangle situation above we add either of the equivalent following constraints:

$$\sum_{e \in S} X_e \leq \lfloor \frac{|S|}{2} \rfloor, \ \forall S \subseteq V, |S| \textit{ is odd,}$$

or:

$$\sum_{e \in \delta S} X_e \geq 1, \ \forall S \subseteq V, |S| \textit{ is odd.}$$

So, the LP relaxation is:

$$\text{maximize} \sum_{e \in E} X_e w_e$$

$$\sum_{e \in \delta(v)} X_e = 1 \qquad \forall v \in V \tag{2}$$

$$\sum_{e \in \delta(S)} X_e \geq 1, \qquad \forall S \subseteq V, |S| \text{ is odd} \tag{3}$$

$$X \geq 0 \tag{4}$$

Define P_M as before and let $P_{LP} = \{\mathbf{X} | \mathbf{X} \text{ satisfies (2), (3), (4)}\}$.

Theorem 9 *Let G be a graph that has a perfect matching, then $P_{LP} = P_M$.*

Proof: Again $P_M \subseteq P_{LP}$ is obvious. We prove the converse by induction on the number of edges. The base $|E| = 1$ is obvious (since it induces a perfect matching itself). Assume $P_{LP} \nsubseteq P_M$. Similar to Claim 7 let $\mathbf{X}$ be a point in P_{LP} that is not in P_M and has the smallest support. Again, we can assume every vertex has degree at least 2. If the graph contains an even cycle similar to proof of Claim 7 we can reduce support size of $\mathbf{X}$.

Now, assume there is no vertex of degree at least 3. So, the support graph is collection of odd cycles. But this contradicts constraint 4. Therefore, some vertex has degree at least 3. Which means $|E| > |V|$. Since $\mathbf{X}$ is a corner point we have at least $|E|$ tight constraints. There are only $|V|$ constraints of type 2, so, $|E| > |V|$ implies one of the tight constraints is of type 3. So, there is a set S with odd cardinality, so that $|S| \geq 3$ and

$$\sum_{e \in \delta(S)} X_e = 1, \ |S| \textit{ is an odd number} \geq 3.$$

Now, let G' and G'' be the graph constructed from G by shrinking S and $\overline{S} = V \setminus S$ into one single node v' and v'', respectively. Let $\mathbf{X}'$, $\mathbf{X}''$ be defined by restricting $\mathbf{X}$ to edges in G' and G'', respectively. Since, $\sum_{e \in \delta(v)} X'_e = \sum_{e \in \delta(S)} X_e = 1$, $\mathbf{X}'$ is a point in P_{LP} for G'. The same is true for $\mathbf{X}''$.

By the induction hypothesis, we can write X', and X'' as sum of sets of perfect matchings $\mathbf{X}_{M_i^1}$, and $\mathbf{X}_{M_j^2}$, respectively. Let L be the common denominator:

$$\mathbf{X}' = \frac{1}{L} \sum_i \mathbf{X}_{M_i^1}$$

$$\mathbf{X}'' = \frac{1}{L} \sum_i \mathbf{X}_{M_i^2}.$$

Since, X', and X'' have same values on $\delta(S)$, after multiplying by L, we can find one to one correspondence between M_i^1 and some $M_{i'}^2$ so that they use the same edge in $\delta(S)$. Let M_i be the union of M_i^1 and $M_{i'}^2$. Then we have:

$$\mathbf{X} = \frac{1}{L} \sum_i \mathbf{X}_{M_i}.$$

$\square$

Comments: LP formulation introduced for general graphs has exponentially many constraints, however using separating oracle techniques one can find the solution in polynomial time.

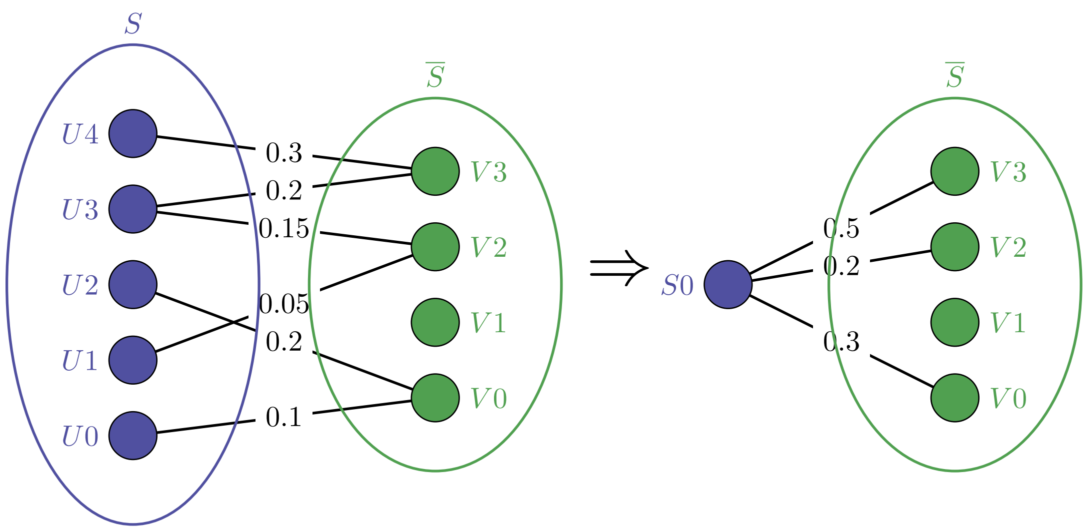

Figure 1: A graph that crossing edges sum to 1 (Left), Graph after shrinking vertices of S into one single vertex (Right).