MS&E 319: Matching Theory
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1 Stable Matching

This lecture is on the stable matching problem. Consider a community with a set of n men, M, and a set of n women, W. Each man m has a ranking of women representing his preferences i.e., if woman w comes before woman w' in m's list, it means that m prefers to marry w rather than w'. Similarly each woman w has a ranking of her preferred men. The *stable marriage* problem asks to pair (match) the men and women in such a way that no two people prefer each other over their matched partners. More formally:

Definition 1 *A* ***matching****, P_M, is a one-to-one mapping from M to W (or equivalently we can define a matching as a one-to-one mapping from W to M).*

Definition 2 *A pair (m, w) is a* ***rogue pair*** *iff*

1. *m and w are not matched with each other in P_M.*
2. *m prefers w to his matched partner $P_M(m)$.*
3. *w prefers m to to her matched partner m' ($w = P_M(m')$),*

Definition 3 *A matching is* ***stable*** *if it doesn't have a rogue pair.*

Example: Assume that we have $M = \{A, B, C\}$ and $W = \{1, 2, 3\}$ with preferences (rankings) given by

$$\begin{array}{rlcrl} A: & 123 & \qquad & 1: & BAC \\ B: & 213 & & 2: & ABC \\ C: & 321 & & 3: & CBA \end{array}$$

Let matching P_M be as follows:

$$P_M(C) = 1 \quad P_M(B) = 2 \quad P_M(A) = 3$$

It is easy to see that the matching above is not stable since $(A, 1)$ is a rogue pair. But the following matching has no rogue pairs ($1, 3, A, C$ get their first choices), and so is stable:

$$P_M(A) = 1 \quad P_M(B) = 2 \quad P_M(C) = 3$$

Question 1 *How can we find a stable matching in general?*

In 1962, Gale and Shapley proposed the Deferred Acceptance Algorithm (1). Here, we assume that each man m (woman w) ranks all the possible women (men), i.e. m's (w's) list is complete.

Algorithm 1 Deferred Acceptance [Gale-Shapley] (men-proposing version)

let all men and women be unmatched.
repeat
  unmatched man m proposes to the most preferred woman w in his list
  if w is unmatched **then**
    match w and m (they become engaged)
  else if w is matched to m', but she prefers m to m' **then**
    match (engage) w to m and leave m' unmatched.
  else
    m removes w from his preference list
  end if
until each m has been matched or has reached the end of his list (i.e., has an empty list)
return matching M of all engaged pairs $(m, P_M(m))$

Claim 4 *The GS algorithm terminates.*

Proof: A man is rejected at most $|W|$ times, each time removing a woman from his preference list. There are $|M|$ such lists, thus it takes at most $|W| \cdot |M|$ steps. $\square$

Claim 5 *The GS algorithm produces a perfect matching whenever $|M| = |W|$.*

Proof: Suppose there is some unmatched man m. Since $|M| = |W|$ there exists some unmatched woman w. Since m is unmatched he has been rejected by *all* women. Since w is unmatched she has never rejected a proposal. This is a contradiction since w is in some position in m's preference list. $\square$

Claim 6 *The matching found by the GS algorithm is stable.*

Proof: Consider pair (m, w) where $P_M(m) \neq w$. According to the GS algorithm, two scenarios are possible:

1. m has proposed to w and was rejected; this means w prefers her current partner to m.
2. m has not proposed to w; this means m prefers his current partner to w.

Neither of the above scenario results in (m, w) being a rogue pair. So there cannot be a rogue pair. $\square$

Theorem 7 *The men-proposing algorithm is* man-optimal*: every man will be matched to the* best *partner he could be matched to in any stable matching.*

Proof: By contradiction. Denote M as the matching produced by Algorithm 1. Consider the **first** event, E^*, that a man is rejected by a woman in the algorithm. Label this man m, and the woman w. m is rejected because w prefers some man m' to m. Suppose there exists some other stable matching $\widetilde{M}$ such that m and w are matched, i.e. $m = P_{\widetilde{M}}(w)$. Suppose m' is matched to w' in $\widetilde{M}$ $(m' = P_{\widetilde{M}}(w'))$. Consider two cases:

Case I: m' **prefers** w **to** w'**,**

$\widetilde{M}$ contains the pairs (m, w) and (m', w'), however (m', w) is a rogue pair in $\widetilde{M}$. This contradicts the assumption that $\widetilde{M}$ is stable.

Case II: m' prefers w' to w,
Going back to the algorithm; m' prefers w' to w but it is matched to w at the time that event E^* happens. This means that m' was rejected by w' before proposing to w. This is contradicting with the assumption that m is the first man that gets rejected in the run of the algorithm. □

Theorem 8 *The men-proposing algorithm is* female-pessimal*: every woman will be matched to the* worst *partner she could be matched to in any stable matching.*

Proof: Again by contradiction. Let M be the output of Algorithm 1. w is matched to m in M. Suppose w prefers m to m' and there exists another stable matching $\widetilde{M}$ in which w is matched to m' and m is matched to another woman w'.

By man-optimality of M (Theorem 7), m prefers w to w'. Matching $\widetilde{M}$ contains (m, w') and (m', w), however, (m, w) is a rogue pair. This contradicts the assumption that $\widetilde{M}$ is stable. □

2 Extensions and Applications

Generalizations to stable marriage present some difficulties. In more realistic scenarios we can expect ranking lists to be incomplete (one may rather be alone than with someone insufficiently qualified), and we should be able to model indifference between subsets of partners. While each of these are solvable in polynomial time separately, solving incomplete lists with indifferent preferences together makes the problem NP-hard.

Another issue of concern is whether there are any incentives to change preference lists. In game theory, a game is called *strategyproof (thruthful)* if players have no incentive to hide information from each other. Could someone state false preferences to gain a benefit from the match? For men, it is trivially truthful, but there is no incentive for women to be truthful! One can design cases where changing the list of preferences will increase the happiness of a given woman with the final matching results.

Furthermore, in the real world, matching with preferences is usually not one to one. Many to one marriage problems, commonly referred to as the college admissions model, are famously employed in several entry level professional labor markets such as the National Residency Matching Program (NRMP). The college admission model assumes some population of students, a number of colleges with certain student capacities, and preferences stated by the colleges and the students.

NRMP has been in effect since the 1950's and has experimented with different matching models to match medical school graduates to hospitals as they enter residency programs. For a long time the system was run as *hospital-optimal*. For this reason, the system was sued for being anti-competitive; the system held up, but was switched from being hospital-optimal to resident-optimal. Interestingly, this switch didn't change the rankings much because the rankings tend to be highly correlated.

Rules for the algorithm used to conduct the matching for NRMP have been updated many times throughout the years as complications arose. One of the major ones was the introduction of couples; that is, an application to a hospital coming not from a single student but from a married couple. Economist Al Roth proved that introducing couples into the match leaves a possibility that no stable match exists.

Exercise 9 *Prove the rural hospitals theorem: if the preferences of men and women are strict, then the set of men and women who are matched is the same in every stable matching.*

Definition 10 *For any two matchings μ and μ', the join operator $\mu \vee \mu'$ is a function that assigns to each man m his more preferred option of the two matches $\mu(m), \mu'(m)$.*

Definition 11 *For any two matchings μ and μ', the meet operator $\mu \wedge \mu'$ is a function that assigns to each man m his less preferred option of the two matches $\mu(m), \mu'(m)$.*

Exercise 12 *If μ and μ' are stable matchings, then $\mu \vee \mu'$ and $\mu \wedge \mu'$ are also stable matchings.*

3 Stable Matching in a Probabilistic Setting

In this section of the notes, we consider the setting in which the preference list of each agent is drawn randomly from some underlying distribution. In particular, we assume that the preference list of each man and each woman is chosen uniformly at random (and independently) from the set of all permutations over potential partners. The goal of this section of the notes is to quantify what it means for the men-proposing algorithm to be man-optimal and woman-pessimal. Exactly how much better off are the men than the women in this setting?

3.1 Balanced Matching Market

Even the slightest imbalance between the number of men and women can have drastic consequences. We first consider the case in which there are n men and n women.

In order to quantify the quality of a stable matching for men and for women, it is crucial to define an appropriate metric. Let R_{men} denote the average rank of a matched man and let R_{women} denote the average rank of a matched woman. With these definitions, we can now state a main result:

Theorem 13 *In the men-proposing algorithm, we have that*

$$(*) \quad R_{men} = \log n$$
$$(**) \quad R_{women} = \frac{n}{\log n}.$$

Example: In a market with 1000 men and 1000 women, men are matched (on average) with their $\log 1000 \approx 7$-th choice woman, while women are matched (on average) with their $\frac{1000}{\log 1000} \approx 145$-th choice man. This example serves to illustrate the striking disparity between the quality of the matching for men and for women.

Proof of $(*)$**.** We will show that $R_{men} = \log n$. Notice that it is sufficient to prove that the GS algorithm terminates after $n \log n$ proposals, as this implies that each of the n men must make $\log n$ proposals (on average). Recall that the algorithm stops iff all women are married and a woman is married iff she has received a proposal. So, how long does it take for all women to have received a proposal? This reduces to an instance of the Coupon Collector's Problem. To this end, let X_i be a random variable denoting the number of proposals until the i-th woman is proposed to. Notice that X_i can be modeled as a geometric random variable with parameter $p_i = \frac{n-i+1}{n}$. As such, it follows that $\mathbb{E}[X_i] = \frac{1}{p_i}$ and $Var(X_i) = \frac{1-p_i}{p_i^2}$. Additionally, let $X = \sum_{i=1}^{n} X_i$. Notice that X is a random variable representing the number of proposals until all women have been proposed to. From here, notice that

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n \sum_{i=1}^{n} \frac{1}{i} = nH_n \approx n \log n,$$

where H_n denotes the n-th Harmonic number.

We now want to show that X is concentrated around its expectation. To this end, we claim that $Var(X) \leq 2n^2$. To see that this is true, notice that

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \frac{1 - \frac{n-i+1}{n}}{\left(\frac{n-i+1}{n}\right)^2} = \sum_{i=1}^{n} \frac{(i-1)n}{(n-i+1)^2} = n \sum_{i=1}^{n} \frac{n-i}{i^2} \leq n^2 \sum_{i=1}^{n} \frac{1}{i^2}.$$

It can be shown that $\sum_{i=1}^{n} \frac{1}{i^2} \leq 2$, which yields the result. From here, we can use Chebyshev's inequality to see that

$$\mathbb{P}\left(|X - n \log n| \geq n \log \log n\right) \leq \frac{2n^2}{(n \log \log n)^2} = \frac{2}{(\log \log n)^2}.$$

Crucially, notice that $\mathbb{P}\left(|X - n \log n| \geq n \log \log n\right) \to 0$ as $n \to \infty$. Furthermore, by Markov's inequality, we have that

$$\mathbb{P}\left(X \geq \frac{3}{2} n \log n\right) = o(1).$$

To summarize, we have that (with high probability) X is concentrated around its mean of $n \log n$, meaning that R_{men} is concentrated around its mean of $\log n$.

$\square$

Proof of $(**)$**.** Next, we will show that $R_{women} = \frac{n}{\log n}$. Fix a woman w and let $\nu(w)$ denote the number of proposals received by w. Notice that $\mathbb{E}[\nu(w)] = \log n$ from the previous analysis. Now, recall that during each iteration of the algorithm, w can only become better off. That is, if she is proposed to by a man she prefers to her current match, she can jettison her current man in favor of the new one. As such, w receives her top choice out of the $\nu(w)$ men that propose to her. This begets the following question: what is the minimum of $\nu(w)$ numbers chosen uniformly at random from 1 to n? The answer is $\frac{n}{\nu(w)+1}$, in expectation. Consequently, we have that

$$\mathbb{E}[R_{women}] = \mathbb{E}\left[\frac{1}{n} \sum_{w \in W} \frac{n}{\nu(w)+1}\right] = \mathbb{E}\left[\frac{n}{\nu(w)+1}\right] \geq \frac{n}{\mathbb{E}[\nu(w)]+1} = \frac{n}{\log n + 1} \approx \frac{n}{\log n},$$

by Jensen's inequality. Also, notice that $\frac{1}{n}\sum_w \nu(w) = R_{men} \leq \frac{3}{2}\log n$ (with high probability), which follows from the previous analysis. This implies that

$$\mathbb{P}(\nu(w) \geq 3\log n) \leq \frac{1}{2}.$$

As such, we have that R_{women} is concentrated around its mean of $\frac{n}{\log n}$. $\square$

3.2 Unbalanced Matching Market

We now consider the situation in which the number of men is not equal to the number of women. For simplicity, we consider the case in which there are $n+1$ men and n women. We now state the main result:

Theorem 14 *Even in the men-proposing algorithm, we have that*

$$(***) \quad R_{men} \geq \frac{n}{1.01\log n}$$
$$(****) \quad R_{women} \leq 1.01\log n.$$

We do not formally prove this theorem here, but we note the following:

Definition 15 *The **core** of a two-sided market is the set of stable matchings.*

In [1], Ashlagi *et al.* show that the competition resulting from even the slightest imbalance yields an essentially unique stable matching, obviously implying a small core.

References

[1] I. Ashlagi, Y. Kanoria, J. D. Leshno, Unbalanced Random Matching Markets: The Stark Effect of Competition *Journal of Political Economy*, 125(1):69-98, 2017.